2

NAVAL POSTGRADUATE SCHOOL

Monterey, California

THESIS

LINEAR QUADRATIC GAUSSIAN CONTROLLER DESIGN
USING LOOP TRANSFER RECOVERY
FOR A FLEXIBLE MISSILE MODEL

by

Fernando Jiménez

December, 1993

Thesis Advisor: Roberto Cristi

Approved for public release; distribution is unlimited.

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave Blank)*

2. REPORT DATE
December 17, 1993

3. REPORT TYPE AND DATES COVERED

4. TITLE AND SUBTITLE
LINEAR QUADRATIC GAUSSIAN CONTROLLER DESIGN USING LOOP TRANSFER RECOVERY FOR A FLEXIBLE MISSILE MODEL

5. FUNDING NUMBERS

6. AUTHOR(S)
Jiménez, Fernando

7. PERFORMING ORGANIZATION NAMES (S) AND ADDRESS(ES)
Naval Postgraduate School
Monterey, CA 93943-5000

8. PERFORMING ORGANIZATION

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Naval Postgraduate School
Monterey, CA 93943-5000

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE
A

13. ABSTRACT *(Maximum 200 words)*

In this thesis, a Linear Quadratic Gaussian Controller (LQG) is designed for a tail controlled surface-to-air missile model in order to meet design specifications. The mathematical model of the flexible missile is subject to uncertainties that may arise from unmodelled dynamics, parameter variation or linearization of nonlinear elements. Since these uncertainties are not taken into account in the LQG controller, μ Analysis is applied to the design in order to evaluate the Robust Performance, Robust Stability, and Nominal Performance of the system. Finally, a Linear Quadratic Gaussian controller is designed using Loop Transfer Recovery (LQGLTR) in order to improve the Robust Stability of the system. It is found that the Robust Stability of the design is improved, but as a consequence of losing nominal performance. The μ Analysis and Synthesis Toolbox and the Control Toolbox of MATLAB were used for the design, assembly, analysis and simulation of the missile flight control system.

14. SUBJECT TERMS
Robust Multivariable Control, Linear Optimal Estimation, Missile Autopilot

16. PRICE CODE

15. NUMBER OF PAGES
67

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UNLIMITED

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. 239-18, 298-102

Approved for public release; distribution is unlimited.

LINEAR QUADRATIC GAUSSIAN CONTROLLER DESIGN USING LOOP TRANSFER RECOVERY FOR A FLEXIBLE MISSILE MODEL

by

Fernando Jiménez
Lieutenant, Peruvian Navy
BSEE, Naval Postgraduate School, 1993

Submitted in partial fulfillment
of the requirements for the degree of

MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

from the

NAVAL POSTGRADUATE SCHOOL
December, 1993

Author: Jiménez
Fernando Jiménez

Approved By: Roberto Cristi
Roberto Cristi, Thesis Advisor

Harold Titus
Harold Titus, Second Reader

Michael A. Morgan
Michael A. Morgan, Chairman
Department of Electrical and Computer Engineering

ABSTRACT

In this thesis, a Linear Quadratic Gaussian Controller (LQG) is designed for a tail controlled surface-to-air missile model in order to meet design specifications. The mathematical model of the flexible missile is subject to uncertainties that may arise from unmodelled dynamics, parameter variation or linearization of nonlinear elements. Since these uncertainties are not taken into account in the LQG controller, μ Analysis is applied to the design in order to evaluate the Robust Performance, Robust Stability, and Nominal Performance of the system. Finally, a Linear Quadratic Gaussian controller is designed using Loop Transfer Recovery (LQGLTR) in order to improve the Robust Stability of the system. It is found that the Robust Stability of the design is improved, but as a consequence of losing nominal performance. The μ Analysis and Synthesis Toolbox and the Control Toolbox of MATLAB were used for the design, assembly, analysis and simulation of the missile flight control system.

TABLE OF CONTENTS

LIST OF FIGURES

I. INTRODUCTION

Highly evasive target maneuver levels and increasingly effective electronic countermeasures push current air defense missile systems to the limits, resulting in more stringent autopilot performance requirements. Conventional autopilot design techniques have worked well in the past, but new design methods are required to obtain improved performance and robustness characteristics from the flight control system in order to satisfy future design specifications. A significant research effort has been directed toward the design and implementation of robust controllers which can guarantee both stability and performance

Robust control design methods such as H_∞ Optimal control or Linear Quadratic Gaussian Loop Transfer Recovery optimize performance and stability based on engineering models which include performance specifications and descriptions of how uncertainty modifies the nominal plant.

Recent advances in robust feedback control, in particular H_∞ control and μ-synthesis, allow for new design methodologies to be applied to the design of missile flight control systems. H_∞ optimal control provides the basis for controller synthesis while mu analysis characterizes performance and stability in the presence of a defined structure for uncertainties. Recent advances in robust control theory offer good prospects for meeting the design needs of next generation missiles. Several anticipated benefits of the robust control design approach are: greater flexibility in the choice of airframe geometry, full use of available airframe maneuver capability and greater tolerance to uncertainty in design models.

Linear quadratic controllers using state estimate feedback are optimal for the nominal model but the performance may be far from satisactory in a real life situation in which the plant differs somewhat from the model. Attractive passband robustness properties of full-state feedback optimal quadratic designs may disappear with the introduction of a state estimator. The Linear-Quadratic-Gaussian with Loop-Transfer-Recovery (LQG/LTR) methodology provides an integrated frequency domain and state approach for design of multi-input multi-output (MIMO) control systems. The advantages of the methodology lie in its ability to directly address design issues such stability robustness and the trade-off between performance and allowable control power.

In this study the problem of the design of a pitch plane autopilot to track normal accelerations commanded from the guidance system for a tail controlled flexible missile [Ref. 1] using Linear Quadratic Gaussian techniques is presented.

In general when a missile is flying at an angle of attack alpha, lift is developed. This lift may be represented as acting at the center of pressure of the structure. The missile will be statically stable or unstable without corrective tail deflections depending on the location of the center of pressure relative to the center of mass.

The control problem requires that the autopilot generate the required tail-deflection delta to produce an angle of attack, corresponding to a maneuver called for by the guidance law, while stabilizing the airframe rotational motion. The primary objective of the missile autopilot is to control the missile's flight path while meeting the specified performance requirements and to mantain the specified stability margins of the airframe over the entire flight envelope, all in the presence of parameter variations, unmodeled dynamics and disturbances. The missile flight control system must guarantee stability and performance in the face of large aerodynamic uncertainties and disturbances. This requires the feedback controller to maintain system stability and loop performance for all variations in the plant behavior. The autopilot design goals are simply to make the missile respond as fast as possible while maintaining stability. Stability robustness goals include robustness to neglected bending dynamics and uncertainties in the aerodynamic parameters.

A pitch plane model describing the longitudinal dynamics of the missile is available for this design. Reasonable accurate mathematical models of the rigid body transfer functions from tail-deflection to the sensor outputs are available too. Sensor measurements for feedback include missile rotational rates (from a rate gyro) and normal acceleration (from an accelerometer).

The sensors are assumed to be subjected to noises, dynamic characteristics, calibration errors and drifts.

The mathematical model of the flexible missile is subject to uncertainties thatmay arise from unmodeled dynamics, parameter variation, linearization of nonlinear elements. Model perturbations or uncertainties such as aerodynamic stability coefficient variation uncertainties and actuator perturbations are considered in the model. Multiplicative parameter uncertainties are considered upon the pitching moment and normal force aerodynamic stability derivatives with respect to angle of attack.

In general there are other uncertainties associated with a missile system that can be considered but are not part of this specific model. For example, the missile may be constantly subjected to environmental disturbances such as wind gust. The wind may induce a sideslip excursion and affect the measurements.

The problem of structural flexibility is also taken into account since this is a long and slender missile. This introduces problems relating to the elastic nature of the airframe. As the autopilot bandwidth increases, coupling between the flight control system and the flexible dynamics of the airframe may take place which limits the achievable airframe response and possibly destabilizes the system. Coupling occurs between the elastic modes and the control system as the control system gyros and accelerometers sense the flexure motion and the rigid body motion. The coupling arises from the fact that the attitude and rate gyros and accelerometers sense both the rigid body changes and the body bending motion.

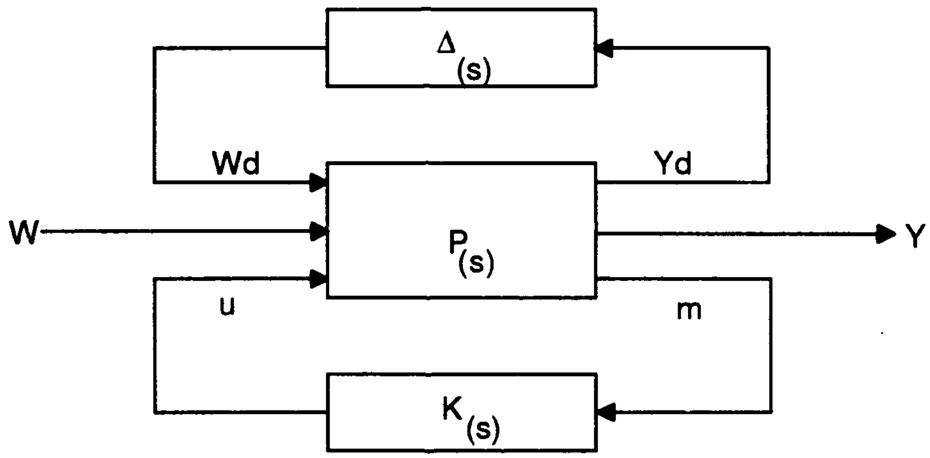
The controller must avoid saturating the tail deflection actuator rate capabilities and destabilizing high frequency flexible body modes of the missile. The push for a smaller time constant to achieve a rapid command response is in conflict with the requirement for robust stability against high frequency uncertain dynamics. During flight, aerolastic forces also act on the airframe, causing high-frequency vibratory modes. The vibrations affect both the acceleration and the pitch rate but only the second one is considered in the model of the missile.

This work is organized as follows. Robust multivariable control theory as well as the Structured Singular Value will be introduced in Chapter II. The Structured Singular Value will play an important role in the evaluation of the missile's performance in Chapter IV. Linear Quadratic Gaussian techniques will be applied to the missile model in Chapter III to yield the desired specifications. This chapter also presents the missile and airframe models used in the design. A technique known as μ analysis is applied in Chapter IV in order to evaluate Robust Performance, Robust Stability and Nominal Performance of the Linear Quadratic Gaussian design for the model of the missile. Linear Quadratic Gaussian with Loop Transfer Recovery is applied in Chapter V to improve the stability robustness of the system. The final chapter summarizes the results and suggest further research.

II. BACKGROUND

A. GENERAL FEEDBACK PERTURBATION MODEL

Any general feedback system in the presence of uncertainties can be represented by the general Δ-P-K framework shown in Figure 1, where the plant model $P_{(s)}$ has been expanded to include additional input and output variables.



P : Plant Model
Δ : Uncertainty
K : Feedback Compensator
W : Disturbance Input
Wd : Feedback through uncertainty
Yd : Output Leading to feedback uncertainty
Y : Reference output

Figure 1. General Δ-P-K Framework.

The additional input Wd includes those plant variables which are subjected to plant parameter variations, while the additional output Yd denotes plant variables upon which those plant variations may act. The matrix $\Delta(s)$ represents a transfer function matrix describing the effect of plant uncertainties. Any linear time invariant control design problem can be arranged into this form [Ref. 2]. The convention adopted here is to normalize exogenous inputs W, outputs Y so that the uncertainty Δ is of a magnitude less than 1. This requires that all scalings be included in the plant description P.

The plant is then modeled as having three inputs and three outputs and is completely described by the transfer function matrix.

$$\begin{bmatrix} Yd \\ Y \\ m \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} Wd \\ W \\ U \end{bmatrix}, \tag{1}$$

or, in a compact form,

$$\begin{bmatrix} Yd \\ Y \\ m \end{bmatrix} = P \begin{bmatrix} Wd \\ W \\ U \end{bmatrix}. \tag{2}$$

In a notation commonly used the generalized plant can be expressed as

$$P_{(s)} = C(sI - A)^{-1} D = \begin{bmatrix} A & \vdots & B \\ \cdots & \cdots & \cdots \\ C & \vdots & D \end{bmatrix}, \tag{3}$$

where the matrices A, B, C, and D are given in the state space model of the plant [Ref. 1].

B. UNSTRUCTURED UNCERTAINTY MODEL

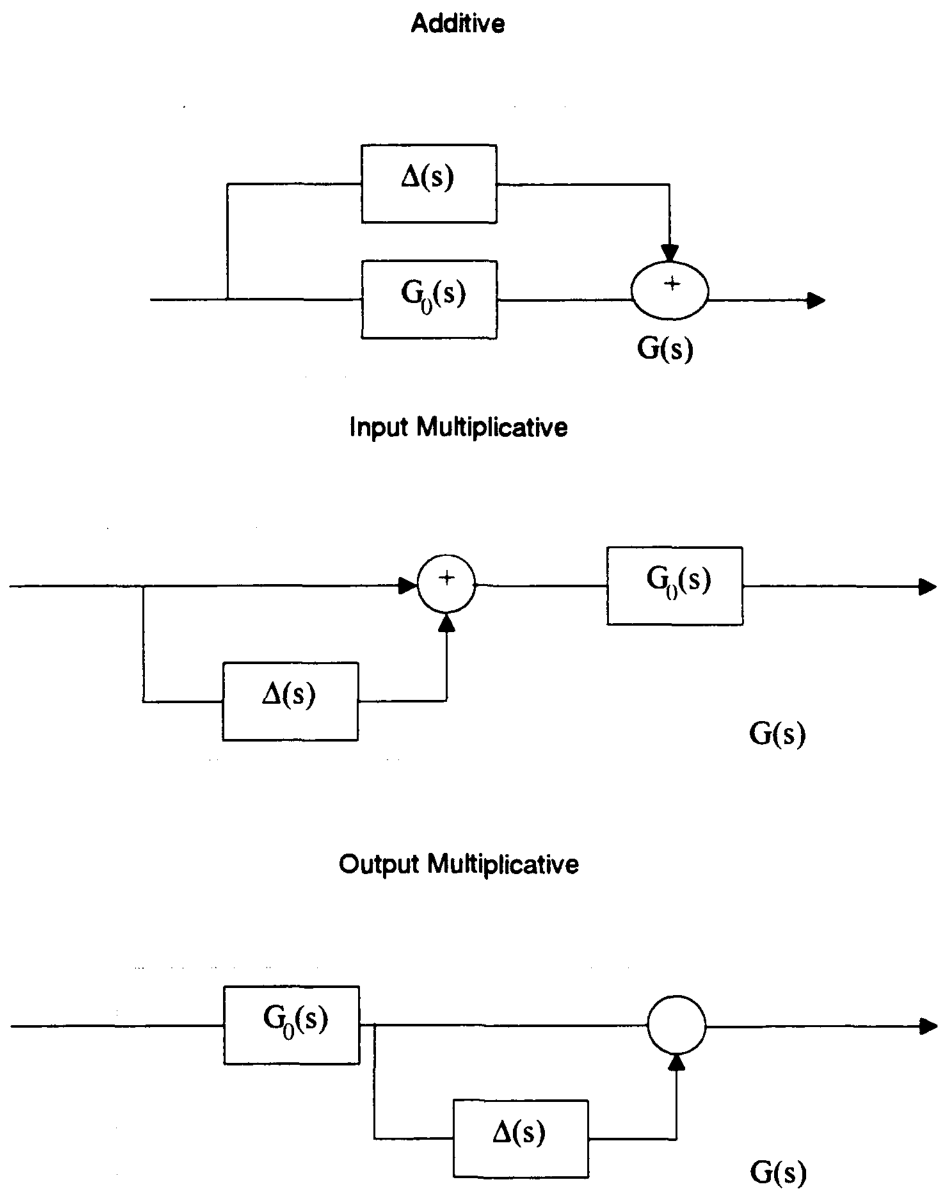
A linear model of a multivariable linear system includes a description of the uncertainties and their structure. The uncertainties are due to parameter variations, unmodeled dynamics, neglected flexible body dynamics, non-linearities, and various assumptions that affect the model of the real system [Ref. 1].

In general, uncertainties in dynamic models could enter the system in a variety of ways, such as parametric or non-parametric, structured or unstructured, stochastic or deterministic.

An unstructured uncertainty is modeled as an unknown with a bounded transfer function incorporated in series or parallel with the plant [Ref. 3]. This type of model of the plant uncertainty is termed unstructured since no detailed model of the perturbation is employed. Figure 2 shows the unstructured uncertainty models that will be considered.

The actual plant will equal the nominal plant for each of these cases when the perturbation equals zero. For evaluation of stability, robustness or performance robustness these perturbations are assumed to be contained within bounds.

$$\|\Delta_i(s)\| \leq \Delta_{MAX} \tag{4}$$



where
$G_0(s)$ represents the nominal plant
x(s) represents a perturbation

Figure 2. Unstructured Uncertainty Models.

C. STRUCTURED UNCERTAINTY MODEL

Structured uncertainties refer to plants which contain one or more uncertain parameters; they also refer to plants which contain both uncertain parameters and unstructured uncertainties.

Uncertain blocks and parameters from various plant locations can be rearranged as shown in Figure 3 [Ref. 3].

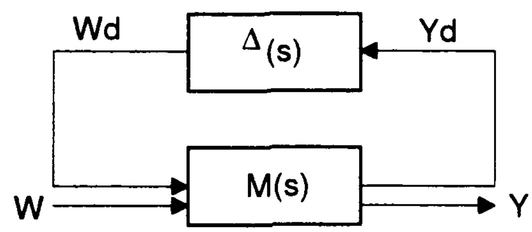


Figure 3. Closed Loop System with Feedback Perturbation.

In this figure M(s) is the combination of the nominal plant P(s) with a feedback compensator K(s), and it represents the dynamics of the nominal closed loop system. It can be divided into 4 blocks as shown below [Ref. 1].

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \tag{5}$$

relative to the inputs and outputs

This matrix partition is called Linear Fractional Transformation and is derived from the following linear equations.

$$\begin{bmatrix} Yd \\ Y \end{bmatrix} = M \begin{bmatrix} Wd \\ W \end{bmatrix} \tag{6}$$

The term M_{22} may be viewed as the nominal closed loop dynamics, assuming no uncertainties and Δ as a linear-fractional uncertainty. The matrices M_{11}, M_{12}, M_{21}, F (M, Δ) describe how Δ affects the overall dynamics, expressed by the transfer function matrix

$$F(M, \Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12} \tag{7}$$

We assume independence of the terms in the structured perturbation which is of the form

$$\Delta(s) = \begin{bmatrix} \Delta_1(s) & 0 & \cdots & 0 \\ 0 & \Delta_2(s) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & \Delta_p(s) \end{bmatrix} \tag{8}$$

or, more compactly

$$\Delta = \text{diag}\{\Delta_1, \Delta_2, \ldots \Delta_p\} \tag{9}$$

The individual uncertainties, Δ_i, that make up the Δ block may have one of several representations [Ref. 1]:

1. Scalar Block; uncertainty appears only once in the real system, such as parameter uncertainty.
2. Repeated Scalar Blocks; where the uncertainty appears multiple time in real system.
3. Full Blocks; example of which includes multivariable neglected dynamics and the performance block.

As in the standard model for unstructured uncertainties, the structured uncertainties will all be scaled so their infinity norms equal one [Ref. 3]

$$\|\Delta\|_\infty = \|\Delta_1\|_\infty = \ldots = \|\Delta_p\|_\infty = 1. \tag{10}$$

where we define the infinity norm as $\|\Delta\|_\infty = \sup_w \bar{\sigma}[\Delta(j\omega)]$.

D. STRUCTURED SINGULAR VALUE

1. Singular Value Decomposition

Any $m \times n$ matrix can be factored as

$$A = V \Sigma U^T \tag{11}$$

where V is an $m \times m$ orthogonal matrix, U is an $n \times n$ orthogonal matrix, Σ is an $m \times n$ matrix of the special form

$$\Sigma = \begin{bmatrix} D & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix} \tag{12}$$

where

$$D = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_k \end{bmatrix} \tag{13}$$

is diagonal.

The elements $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k > 0$ are all positive real, and are called the singular values of the matrix A. The singular decomposition has the property that the L_2 gain of the matrix lies between the smallest and the largest singular values.

$$\underline{\sigma} = \sigma_p \leq \frac{\|Mx\|}{\|x\|} \leq \sigma_1 = \bar{\sigma} \tag{14}$$

where $\underline{\sigma}$ is the smallest singular value and $\bar{\sigma}$ is the largest singular value.

For SISO systems the concept of "gain" is synonymous with the classical Bode magnitude plot. This Bode magnitude is equal to the amount of amplification of a sinusoidal input by the system.

For multivariable systems, the extension of the Bode gain is the singular value of a transfer matrix M evaluated at a frequency upon the imaginary axis, $M(j\omega)$ [Ref. 4].

If we consider the transfer function matrix M evaluated at any frequency point $M(j\omega)$, the largest amplification that a sinusoidal signal of frequency ω will receive as it passes through M is equal to the maximum singular value of M at $s = j\omega$.

2. The Principal Gains

The principal gains are the frequency dependent singular values of the system. They can be used to assist the designer in determining the robustness of the multivariable system and are a good indicator of the system's ability to cope with disturbance rejection.

The principle gains will be denoted as [Ref. 5]

$$\sigma(G(j\omega) = \gamma \tag{15}$$

where $G(j\omega)$ is the frequency dependent transfer function of the system.

If the principal gains of the transfer function matrix of the MIMO system are ordered such that

$$\gamma_1 < \gamma_2 < \ldots < \gamma_n, \tag{16}$$

then γ_1 and γ_n have the property that they bound the L_2 gain between the disturbance input vector $d(s)$ and the output vector $y(s)$

$$\gamma_1 \leq \frac{\|Y(s)\|}{\|s(s)\|} \leq \gamma_n \tag{17}$$

Thus if γ_n is determined for the transfer function matrix then this is the maximum gain which can be obtained by any disturbance vector $d(s)$ and gives a worst case indication of the disturbance rejection properties of the control system while γ_1 gives an indication of the best rejection by the system.

Large values of σ_{MIN} over a large range of frequencies result in good tracking of reference inputs and good rejection of disturbances.

The maximum gain of the system will be denoted by

$$\bar{\sigma}(G(j\omega)) \tag{18}$$

and the minimum gain by

$$\underline{\sigma}(G(j\omega)) \tag{19}$$

The maximum gain that a stable linear system M can deliver is then the supremum of the maximum singular values taken over all frequencies.

3. The Infinity Norm

Mathematically, the infinity norm of a transfer function matrix M is defined as the supremum over all frequencies of the maximum singular value $\bar{\sigma}$ of $M(j\omega)$.

It is denoted by

$$\|M\|_{\infty} = \sup_{w} \bar{\sigma}[M(j\omega)] \tag{20}$$

The first engineering relevance of the infinity norm is that if a control designer seeks to design a controller K for minimizing the transfer of energy or the transfer of power or the transfer of white noise through a closed loop system M which depends on K, then the controller K should be designed so that the infinity norm M, $\|M\|_{\infty}$, is minimized over all controllers which stabilize the closed loop system M.

The second engineering relevance of the infinity norm is that it is essential in establishing robustness of uncertain systems.

4. The Structured Singular Value

The structured singular value of a complex matrix M denoted by $\mu(M(j\omega))$, with respect to the block structured $\underline{\Delta}$, is based on the multivariable Nyquist Theorem and is defined as

$$\mu(M(j\omega)) = \frac{1}{\min\limits_{\Delta \in \underline{\Delta}} [\bar{\sigma}(\Delta) : \ \det(I + M\Delta(j\omega)) \ = \ 0} \tag{21}$$

where $\underline{\Delta}$ denotes the set of all transfer function matrices with the appropriate block diagram form.

It can be shown that the structured singular value is the inverse of the stability margin for multivariable systems [Ref. 3]. In other words, M is the inverse of the smallest magnitude of a destabilizing perturbation of M.

The structured singular value is a measure of robustness of the model, to structured perturbations.

The structured singular value provides an indication of how much uncertainty can be tolerated before the systems becomes unstable. It will be used in Chapter IV in order to evaluate the robust performance and robust stability of the Linear Quadratic Guassian design for the missile model proposed in Chapter III.

III. LINEAR QUADRATIC GAUSSIAN (LQG) DESIGN

In this chapter we apply LQG techniques to the design of a controller for the model of a missile with flexible modes proposed in Section A, in order to meet design requirements specified in Section B.

A. MISSILE MODEL

The airframe model used in this thesis was presented by Bibel [Ref. 1]. A block diagram configuration for the model of the surface launched, antiair homing missile to be considered is shown in Figure 4. The missile is formed basically by the following sub-systems: fin actuator, rigid body dynamics, rate gyro, accelerometer, and flexible body dynamics. The State Space Model representation of this Multi-Input Multi-Output System can be described by

$$\dot{x} = Ax + Bu + \Gamma W, \tag{22}$$

$$y = Cx + Du + \Psi V, \tag{23}$$

where

A = Plant Matrix, B = Input Matrix, C = Output Matrix, D = Feedforward Matrix,

Γ = Input Matrix, Ψ =Measurement Noise Input Matrix.

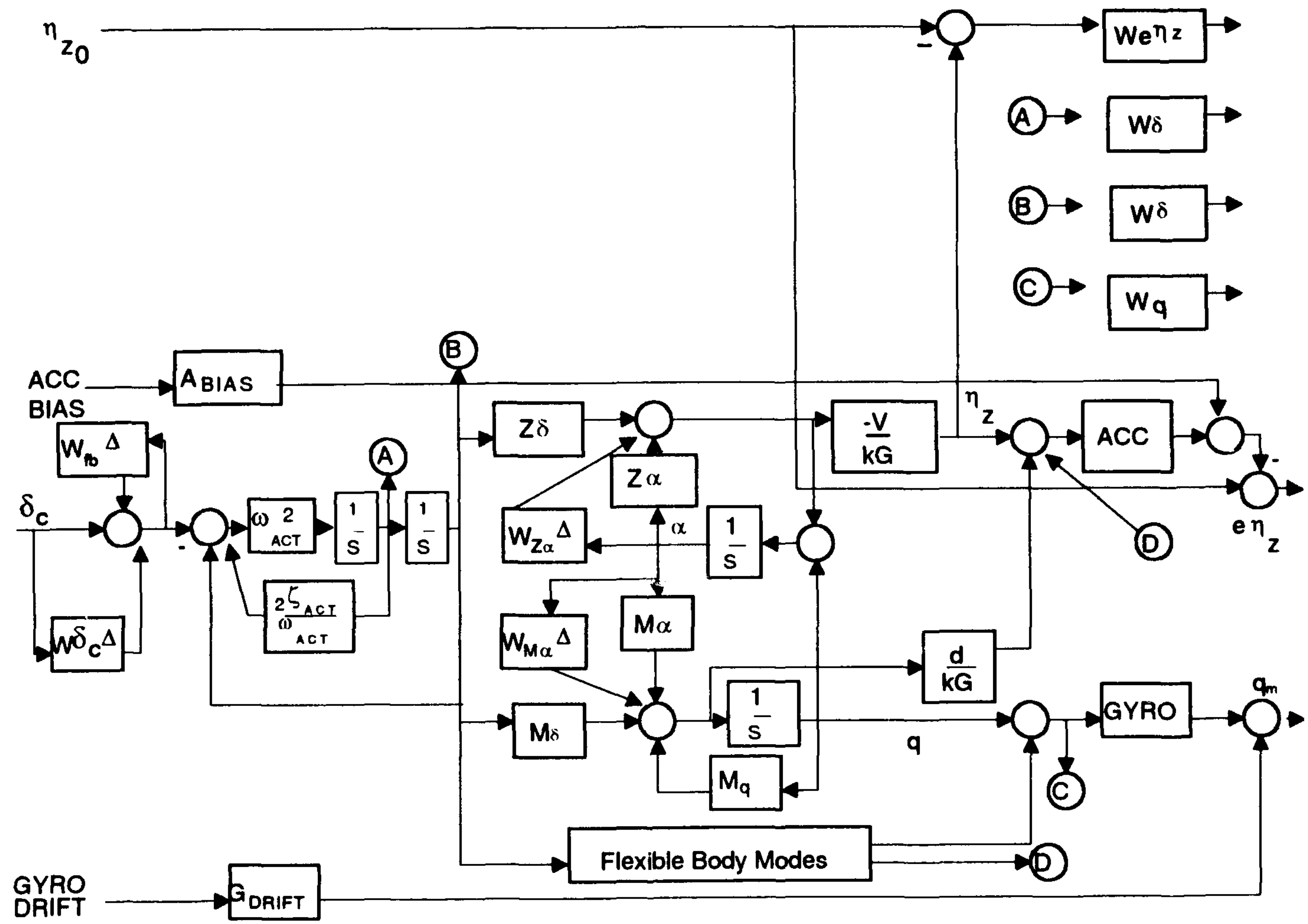


Figure 4. Model of the Tail Control Surface-to-Air Flexible Missile.

The **reference input** to the system is given by the commanded acceleration and the control input is given by the commanded fin deflection. Two **disturbances** entering the system, accelerometer bias and gyro drift, and two **measurements**, measured acceleration error and measured pitch, are modelled as white noise processes with intensities W and V, respectively.

$$V(t) = \begin{bmatrix} V_1(t) & V_{12}(t) \\ V'_{12}(t) & V_2(t) \end{bmatrix} \qquad t \geq t_0 \ , \tag{24}$$

where $V_1(t)$ and $V_2(t)$ are the intensity of W and V, respectively and are correlated. $V_{12}(t)$ is equal to 0 (uncorrelated) and W and V are independent.

We define the covariance matrices as

$$E[W(t)W'(t+\tau)] = Q\delta(t-\tau), \tag{25}$$

$$E[V(t)V'(t+\tau)] = R\delta(t-\tau). \tag{26}$$

For the purpose of modeling these two disturbances, models for biases have been added to the nominal plant model. The error between the commanded acceleration and the measured normal acceleration and the measured body pitch rate are the **feedback signals** to the controller.

The state variables considered in the nominal model are summarized in Table I.

TABLE I. NOMINAL MODEL STATE VARIABLES AND PARAMETERS.		
$\dot{\delta}$	:	Fin Rate
δ	:	Actual Fin Deflection
α	:	Angle of Attack
q	:	Pitch Rate
δ_c	:	Commanded Fin Deflection
η	:	Normal Acceleration (measured by the accelerometer)
η_c	:	Commanded Acceleration
η_{ERROR}	:	Error between η_c and η
η_m	:	Measured Acceleration
q_m	:	Measured Pitch Rate
e_m	:	Measured error in Acceleration

The model to be considered in the design of the linear quadratic gaussian controller combines the nominal plant and the process-noise model but does not include the description of the uncertainties and their structure (i.e., two aerodynamic stability coefficient variation uncertainties, two actuator perturbations, multiplicative parameter uncertainties on the pitching moment, and normal force aerodynamic stability derivatives with respect to the angle of attack).

B. AUTOPILOT DESIGN REQUIREMENTS

The primary objective for the design of the autopilot is to obtain the fastest time response possible while maintaining **stability robustness**. The following time domain specifications are to be considered in the design of the linear quadratic gaussian controller for the missile autopilot.

Settling Time	$Ts < 0.4$ secs	(Time constant = 0.1 secs)
Steady State Error	$e_{ss} = 0.1$ %	(Due to a step command)
Overshoot	$Ov \leq 4\%$	(Due to a step command)

The linear quadratic gaussian controller has to be able to handle the effects of the airframe's elastic dynamics.

C. PROPOSED DESIGN

The design of the linear quadratic gaussian controller for the model of this missile is given by a combination of the solutions of the **Stochastic Optimal Regulator** and **Optimal Estimator** problem.

The optimal control, solution of the stochastic optimal regulator problem, which minimizes the next performance index

$$J = \int_0^\infty y^2(t) + \rho u^2(t) dt, \tag{27}$$

where $y(t)$ represents the output of the accelerometer and $u(t)$ the control input to the system is given by state feedback

$$u = -K_f x \tag{28}$$

$$K_f = R^{-1} B' P \tag{29}$$

where P is the unique positive definite matrix satisfying the Matrix Riccati Equation

$$PA + A'P - PBR^{-1}B'P + Q = 0 \tag{30}$$

provided that Q is symmetric and positive semidefinite, R is symmetric and positive definite, the pair (A,B) is controllable, with the controllability matrix given by

$$\xi = [B \; AB \; A^2B \ldots A^{n-1}B] \tag{31}$$

The scalar ρ plays the role of a Lagrange multiplier. As ρ decreases, the integrated square regulating error decreases, but the integrated square input increases. From simulation, ρ was chosen to be 0.1. The response of the missile due to a unit step commanded acceleration with the Linear Quadratic Regulator is shown in Figure 5.

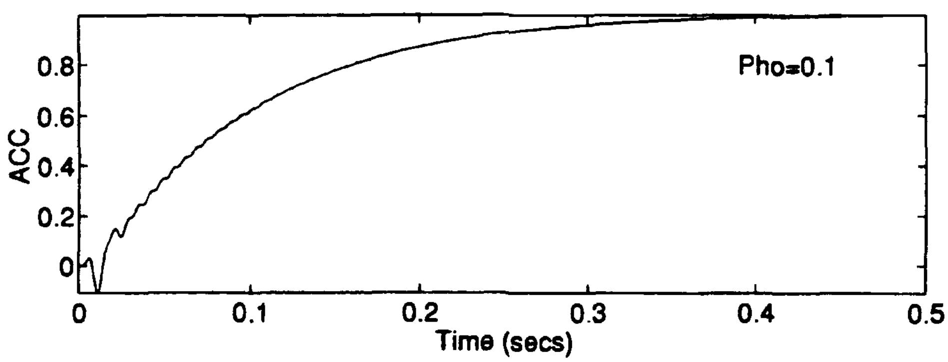


Figure 5. Missile Response Due to a Unit Step Commanded Acceleration.

The solution of the Optimum Observer or State Estimator given by Kalman and Bucy can be expressed by the following Differential Equation

$$\hat{x}' = A\hat{x} + Bu + K_o(y - C\hat{x}). \tag{32}$$

The Kalman Gain K_o is optimally chosen so as to minimize

$$E\{e^T(t)W(t)e(t)\}, \tag{33}$$

and is given by

$$K_o(t) = -P_e(t)C(t)R^{-1}(t); \tag{34}$$

where $P_e(t)$ is the solution of the matrix Ricatti equation

$$\dot{P}_e(t) = P_e(t)F'(t) + F(t)P_e(t) - P_e(t)H(t)R^{-1}(t)H'(t)P_e(t) + Q(t), \tag{35}$$

where

$$e(t) = x(t) - \hat{x}(t), \tag{36}$$

and

$$F_e(t) = F(t) + K_o(t)H'(t). \tag{37}$$

This has a solution provided that

W is symmetric and Positive Semidefinite

V is symmetric and Positive Definite

The pair (A,C) is observable

where the observability matrix is given by

$$\vartheta = [C^T \; A^TC^T \; A^{T^2}C^T \ldots A^{T^{N-1}}C^T] \tag{38}$$

The optimal output feedback control system is represented below. This combines the model of the nominal plant with the optimal feedback gain and the optimal estimator. It also includes a state command matrix N_x that defines the desired value of the state, x_r. Since this is a tracking problem, N_x should transform the reference input r to a reference state. It can be shown that N_x can be calculated from the following matrix equation:

$$\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} \Phi - I & \Gamma \\ Hr & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{39}$$

where Φ and Γ represent the plant matrix and input matrix of the discrete representation of the model of the system.

The Reference State is given by

$$x_r = N_x r \tag{40}$$

and it represents the state we want to track.

Equations (22), (28), (32), and (40) can be combined in order to obtain the State Space Model for the Closed Loop Optimal Output Feedback System shown in Figure 6. From the Control Law given by

$$u = -K_f(\hat{x} - x_r), \tag{41}$$

and substituting Equation (41) into Equation (22) we obtain the closed loop dynamics

$$\dot{x} = Ax - BK_f(\hat{x} - x_r) + \Gamma W. \tag{42}$$

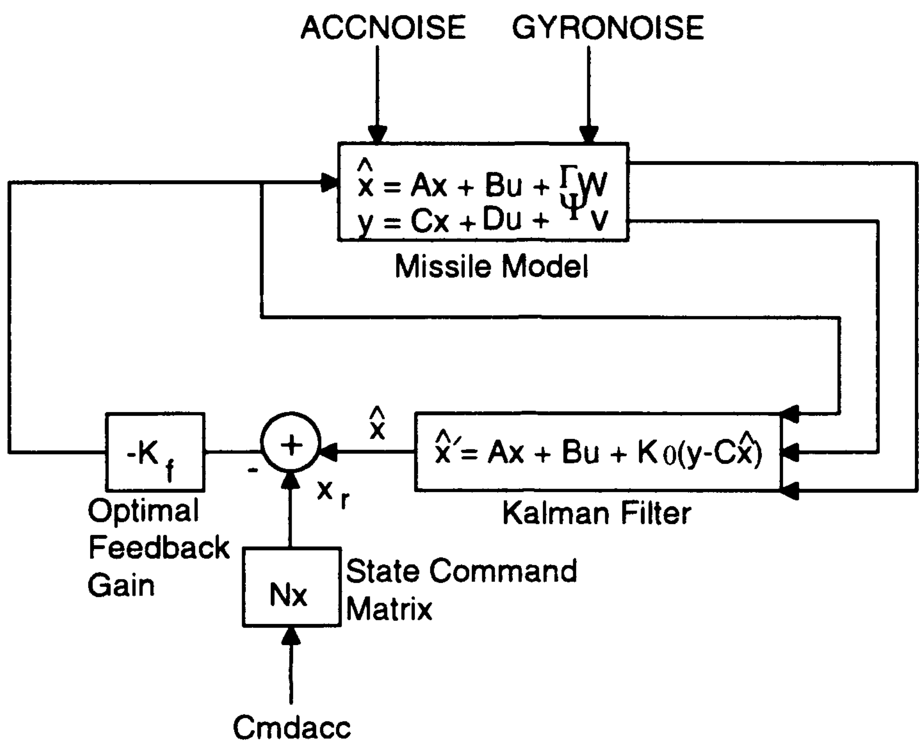


Figure 6. Closed Loop Optimal Output Feedback System.

which can be written as

$$\dot{x} = Ax - BK_f\hat{x} + BK_f x_r + \Gamma W. \tag{43}$$

Combining these equations and performing the required algebra we obtain

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}}' \end{bmatrix} = \begin{bmatrix} A & -BK_f \\ K_oC & A-(B+D)K_f-K_oC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} BK_fN_x & I & 0 \\ (B+D)K_fN_x & 0 & K_o\Psi \end{bmatrix} \begin{bmatrix} r \\ W \\ V \end{bmatrix} \qquad (44)$$

which represents the State Space Model of the Optimal Output Feedback Control System.

The acceleration time response of the closed-loop system to a unit step acceleration command was obtained by computer simulation and is shown in Figure 7.

The missile's angle of attack, body pitch rate, gyro output, commanded fin deflection and fin rate, are shown in Figures 8, 9, 10, 11, and 12.

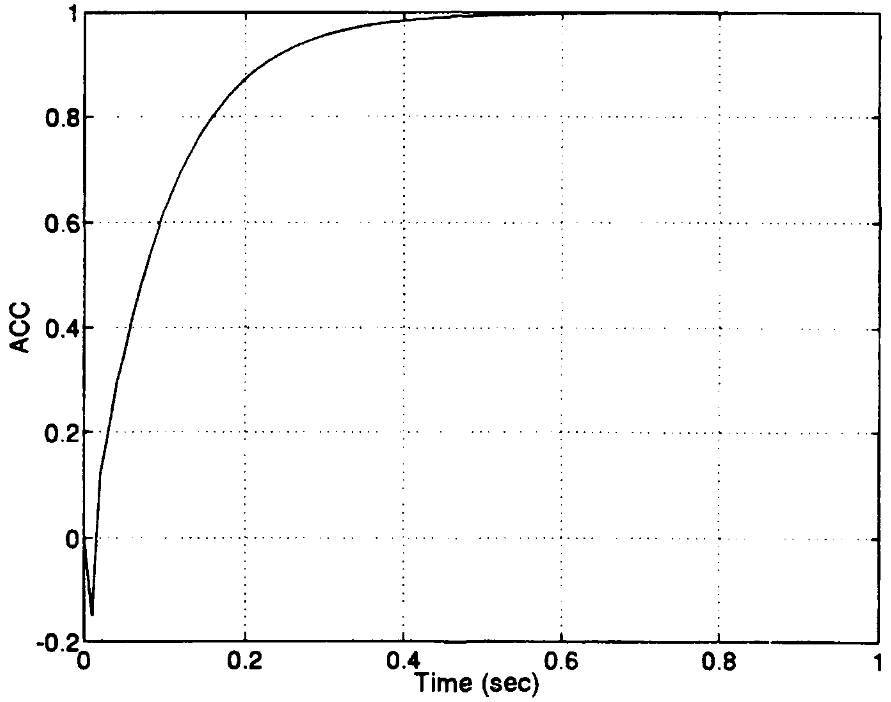


Figure 7. Achieved Normal Acceleration Due to a Unit Step Commanded Acceleration.

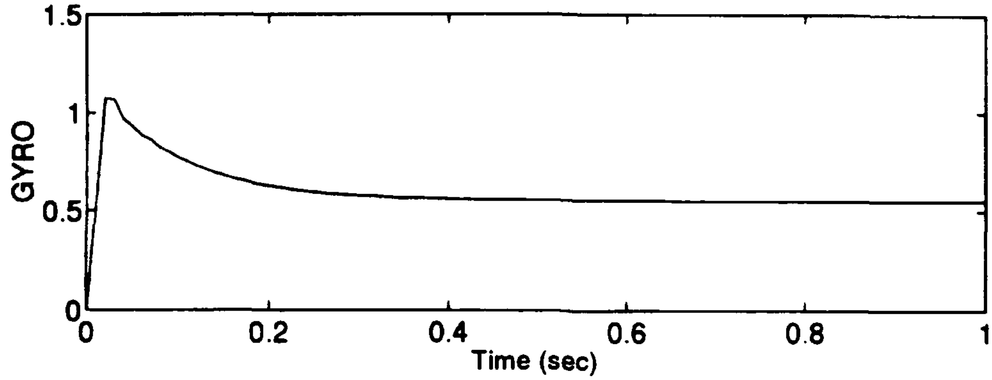


Figure 8. Gyro Output Due to a Unit Step Commanded Acceleration.

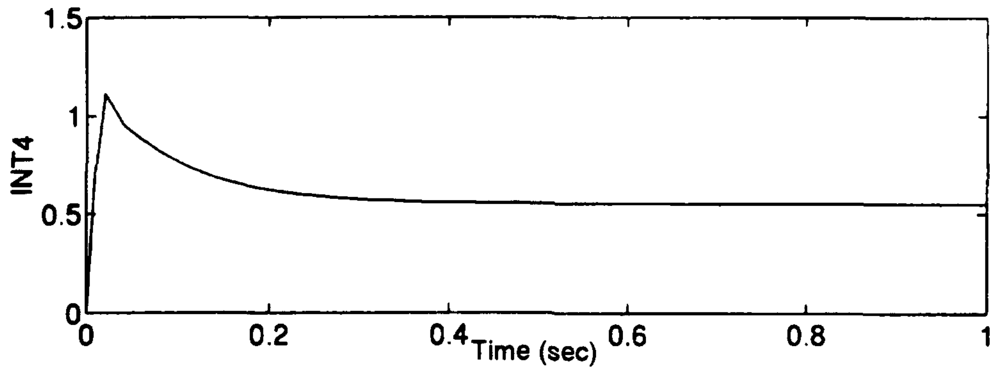


Figure 9. Pitch Rate Due to a Unit Step Commanded Acceleration.

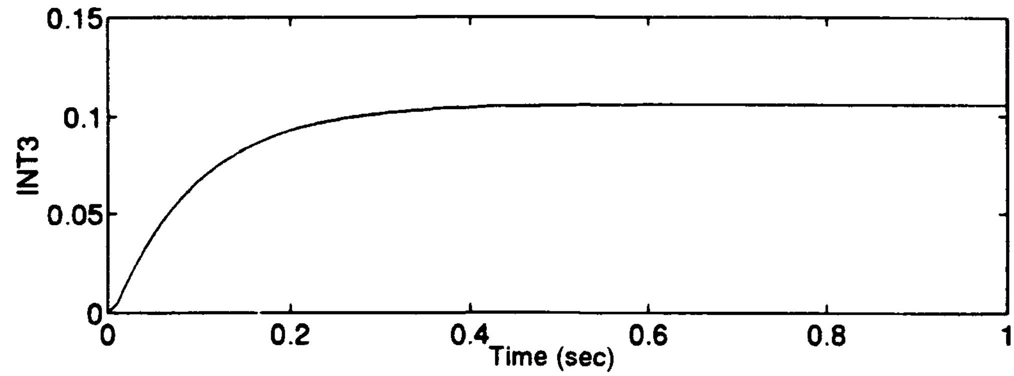


Figure 10. Angle of Attack Due to a Unit Step Commanded Acceleration.

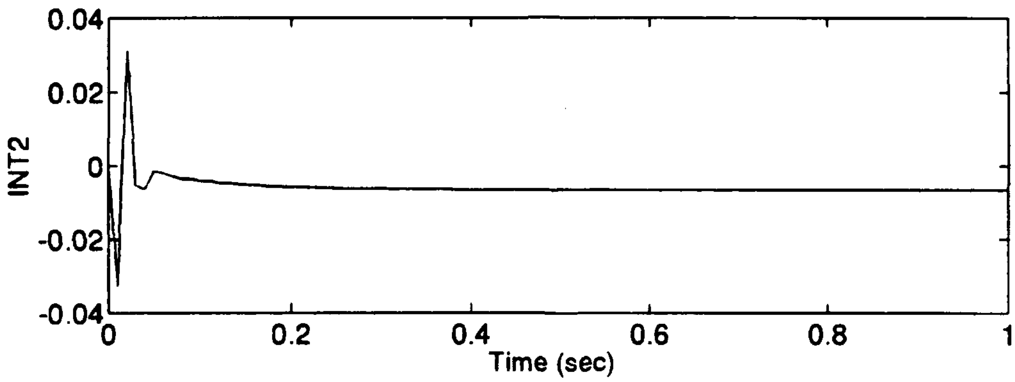


Figure 11. Actual Fin Deflexion Due to a Unit Step Commanded Acceleration.

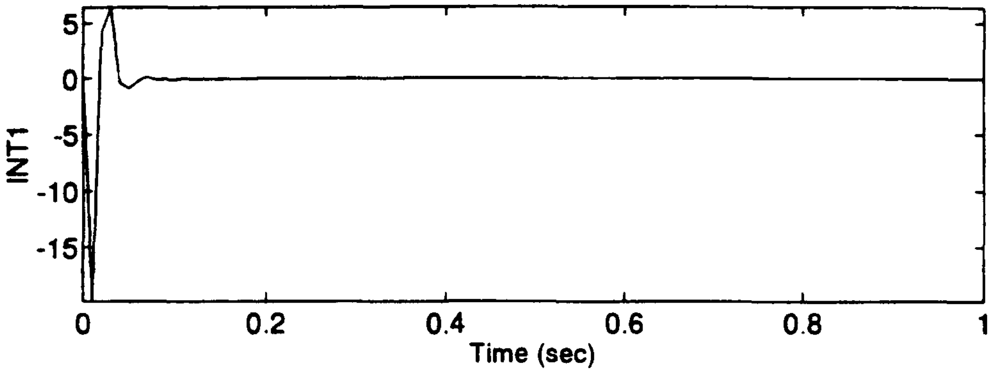


Figure 12. Fin Rate Due to a Unit Step Commanded Acceleration.

The following time domain specifications were achieved with a standard LQG design:

Final Value	:	1.0000	unit
Time to Peak	:	0.4500	secs
% Overshoot	:	0.0000	%
Rise Time	:	0.2500	secs
Settling Time	:	0.4000	secs
Steady State Error	:	0.0000	units

These specifications were obtained from Figure 7. The acceleration response is critically damped and is the fastest that can be obtained for this particular design.

If we observe the missile's angle of attack and body pitch rate responses it can be noticed that these two do not affect the flexible body dynamics. The fin deflection limit of 40 degrees and the fin rate limit of 300 degrees/sec are not exceeded.

So far all of the required specifications for which the controller was designed have been met, but we still have to check the stability of the system. The LQG controller was designed for the nominal plant model of the missile and may or may not meet robust stability requirement.

In the next chapter this design will be tested using a technique called μ-analysis which evaluates the robust performance, robust stability and nominal performance of the design.

IV. μ-ANALYSIS FOR LINEAR QUADRATIC GAUSSIAN DESIGN

The goal of any controller design is that the overall system is stable and satisfies desired performance requirements. These requirements should be satisfied at least when the controller is applied to the *nominal* plant, that is, nominal stability and nominal performance is required. In practice the real plant is not equal to the model. Even though there are uncertainties in the dynamics of any real system we try to quantify the unmodeled dynamics.

In the design of the Linear Quadratic Gaussian controller, the nominal plant model of the flexible missile was assumed to be known exactly. This is rarely the case due to unmodeled dynamics, parameter variations, uncertainties, etc. Even if these uncertainties are neglegible, it may be desirable to design a controller for a plant whose parameters vary deterministically as a consequence of environmental factors or component failures.

When the controller designed for a nominal plant model is implemented on the real system, there are no guarantees on the resulting performance of the system. Even requirements as basic as stability may not be met. The deviation from the expected behavior of the system clearly depends on the accuracy of the model. Because models are never perfect, robustness analysis is necessary to determine the possibility of instability or inadequate performance in the face of uncertainty of the plant dynamics. A controller should guarantee stability, proper response to commands, and reduction of response perturbations caused by disturbance inputs. Ability to perform the first function under parameter variation is called *stability robustness*, while ability to conduct the remaining two functions is termed *performance robustness*.

A. NOMINAL STABILITY AND ROBUST STABILITY

If the LQG controller is included in the nominal plant P, the following class of general matrix transformation called **Linear Fractional Transformation** can be defined as [Ref. 4]

$$F(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \tag{45}$$

F(P,K) denotes the transfer function from $\begin{bmatrix} Wd \\ W \end{bmatrix}$ to $\begin{bmatrix} Yd \\ e \end{bmatrix}$. This particular structure is shown in Figure 13.

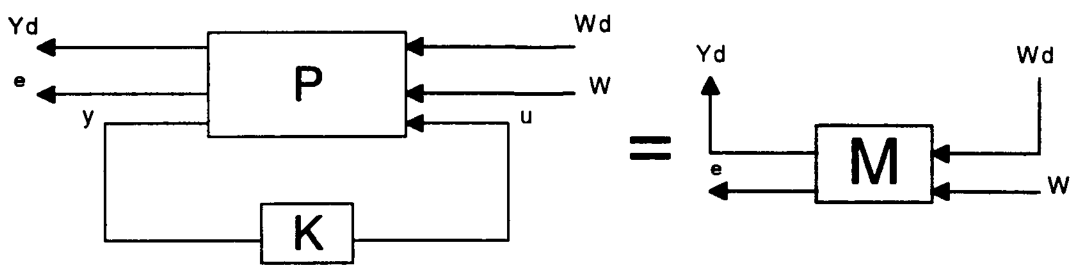


Figure 13. Linear Fractional Transformation.

The complex matrix M, represents the closed loop transfer function between $\begin{bmatrix} Wd \\ W \end{bmatrix}$ and $\begin{bmatrix} Yd \\ e \end{bmatrix}$. It can be partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}. \tag{46}$$

Therefore, the system is nominally stable if M is stable. The stability of a MIMO system is determined by analyzing the feedback system shown in Figure 14. The poles of this feedback system are the solutions of the charateristic equation:

$$\det\{I + M_{11}(s)\Delta(s)\} = 0 \tag{47}$$

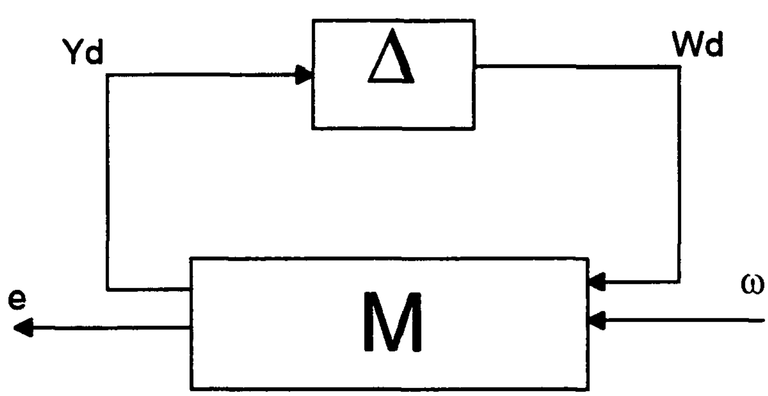


Figure 14. Nominal Closed Loop Plant with a Feedback Perturbation.

It can be shown, by the small gain theorem [Ref. 3], that the system is internally stable if and only if

$$\|M_{11}\|_{\infty}\|\Delta\|_{\infty} < 1 \tag{48}$$

for all possible perturbations. Given that

$$\|\Delta\|_{\infty} \leq 1, \tag{49}$$

the system satisfies the robust stability requirement if

$$\|M_{11}\|_{\infty} \leq 1 \tag{50}$$

when Δ is a full uncertainty block or

$$\text{Max}\mu_{\Delta\infty}(M_{11}(jw))\|\Delta\|_{\infty} \leq 1 \tag{51}$$

if

$$\Delta \varepsilon \bar{\Delta} \quad \text{(Structured Uncertainty)} \tag{52}$$

where μ is called the Structured Singular Value (SSV) defined in Chapter II.

B. NOMINAL PERFORMANCE

The closed-loop system achieves nominal performance if the performance objective is satisfied for the nominal plant model. Mathematically, the system satisfies nominal performance if

$$\|M_{22}\|_{\infty} \leq 1. \tag{53}$$

C. ROBUST PERFORMANCE

A closed-loop system will be said to have robust performance if the total system meets some minimum objectives for all possible perturbations. Given a structured uncertainty

$$\Delta \varepsilon \tilde{\Delta}, \tag{54}$$

where

$$\|\Delta\|_{\infty} \leq 1 \tag{55}$$

and the performance specification

$$\|T(s)\|_{\infty} \leq 1, \tag{56}$$

where

$$y(s) = T(s)W(s), \tag{57}$$

the system will posses "Performance Robustness" if and only if

$$\underset{\omega}{\mathrm{Max}}\ \mu[M(jw)] < 1. \tag{58}$$

To compute the SSV, a block diagonal structured perturbation of the form $\begin{bmatrix} \Delta & 0 \\ 0 & \Delta_p \end{bmatrix}$, where $\Delta \varepsilon \bar{\Delta}$ and Δ_p is unstructured, is assumed.

These are all criteria which quantify the performance and stability margins of a mulitvariable system. In particular, if we plot these as functions of frequency, the system meets all of the specifications, provided they are all less than one. The uncertainties are normalized as in Equation (49). All of this can be shown by using the missile model introduced in Chapter III with the LQG design. Robust performance, robust stability, and nominal performance tests are shown in Figure 15. Notice that only nominal performance is met, since it is less than one, but robust performance and robust stability are not met. This means that we can find a set of perturbations for which the system will not remain stable.

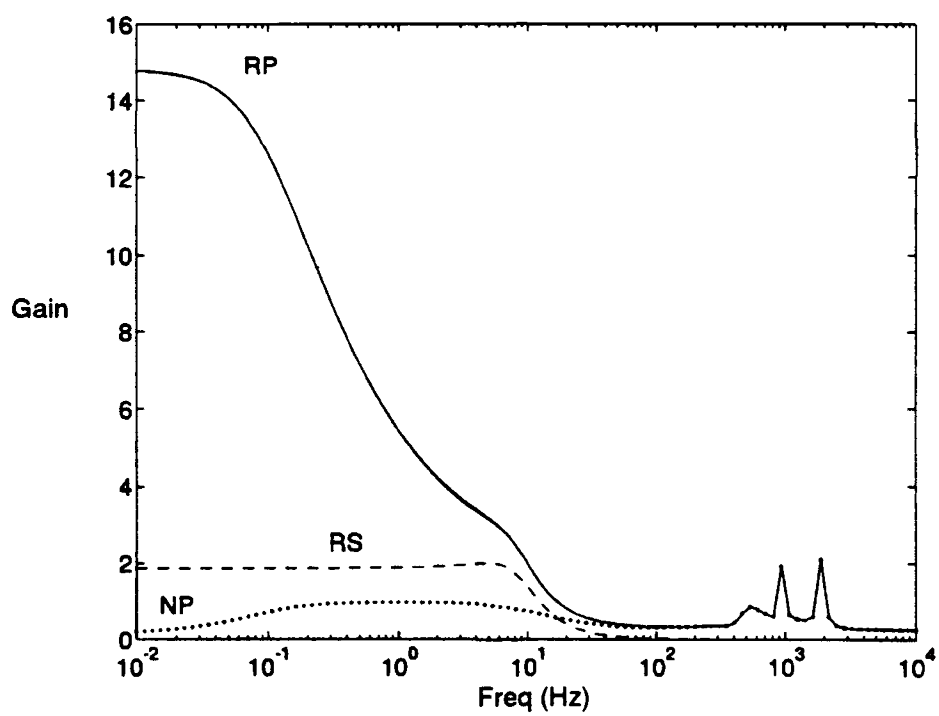


Figure 15. Robust Performance, Robust Stability, Nominal Performance of LQG Design.

The LQG controller meets the design goal at the nominal operating condition. However, it becomes sensitive in the presence of parameter variations.

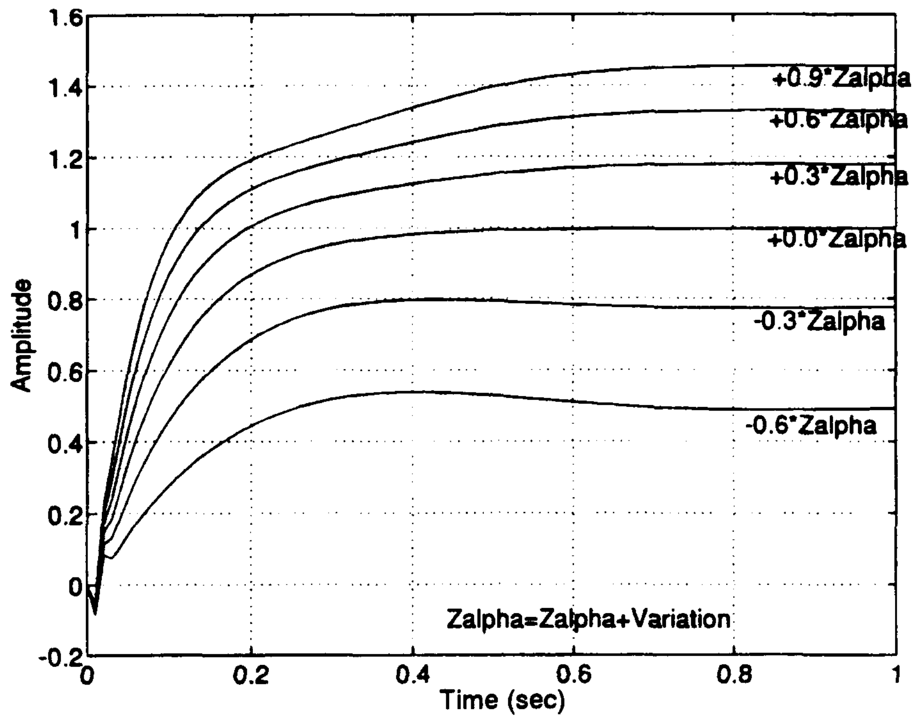


Figure 16. LQG Design Under Uncertainty Variations.

As stated in Chapter III, there are four model perturbations or uncertainties in the design model of the missile. These include two aerodynamic stability coefficient variation uncertainties and two actuator perturbations. One of the desired controller attributes is to remain insensitive to angle of attack changes during flight. Step responses of the closed-loop system with the LQG controller designed in Chapter III for different values of the uncertainty Z_α are shown in Figure 16. It can be noticed that the LQG design meets the specifications for which it was designed when the parameter Z_α remains in its nominal value. Once the parameter Z_α changes, the LQG controlller is unable to track to desired output, and the specifications are no longer met.

The LQG regulator guarantees stability under nominal operating conditions and meets the design specifications for which it was designed. However, stability robustness was not considered explicitly in establishing the optimality criteria. When the estimator is part of the feedback loop of a stochastic regulator, dynamic interactions between the estimator and the controlled system increase the potential for instability.

In the next chapter, a technique known as loop transfer recovery will be used in conjunction with the LQG design in order to improve the robust stability of the system.

V. LQG LOOP TRANSFER RECOVERY DESIGN (LQGLTR)

A major objective of feedback system design is to achieve a *nominal performance* specification for a given design model of the plant, and to *maintain this performance over a range of expected errors between the design model and the true plant.*

As stated in Chapter IV, the attractive passband robustness properties of full-state feedback optimal quadratic designs might disappear with the introduction of a state estimator. However the loop properties of a LQ design can be recovered by a suitable adjustment to the LQG design process with a technique known as **Loop Transfer recovery.**

Given the state space model of the system defined by the matrices A, B, C, and D, the transfer function of the system can be denoted by

$$P(s) = C\Phi B + D, \tag{59}$$

where

$$\Phi = (sI - A)^{-1}. \tag{60}$$

The open loop transfer function at the plant input can be written as

$$L_t(s) = K\Phi B \tag{61}$$

where $L_{T(s)}$ refers to the target loop, and K is the controller gain. By adding *fictitious noise* to the plant input model representing a loose way plant variations, uncertainty, or unmodeled dynamics, the estimator design can be adjusted in such a way that the robustness properties of the LQG design can be recovered.

For this purpose the controller $K(s)$ can be parameterized as a function of a scalar parameter σ in order to obtain a family of controllers $K(s,\sigma)$ where σ; variance of the fictitious noise injected into the input of the plant, represents plant uncertainty, and plays the role of a **tuning parameter** of the optimal observer.

It can be shown that Loop Transfer Recovery (LTR) at the control input is achieved if [Ref. 6]

$$K(s,\sigma)P(s) \rightarrow L_t(s) \tag{62}$$

pointwise in s as $\sigma \rightarrow \infty$.

The exact design procedure **depends on the point where the unstructured uncertainties are modeled** and **where the loop is broken** to evaluate the open-loop transfer matrices. Commonly either the input point or the output point of the plant is taken as such a point. The design which was proposed in Chapter III is shown with the loop broken at the control input in Figure 17.

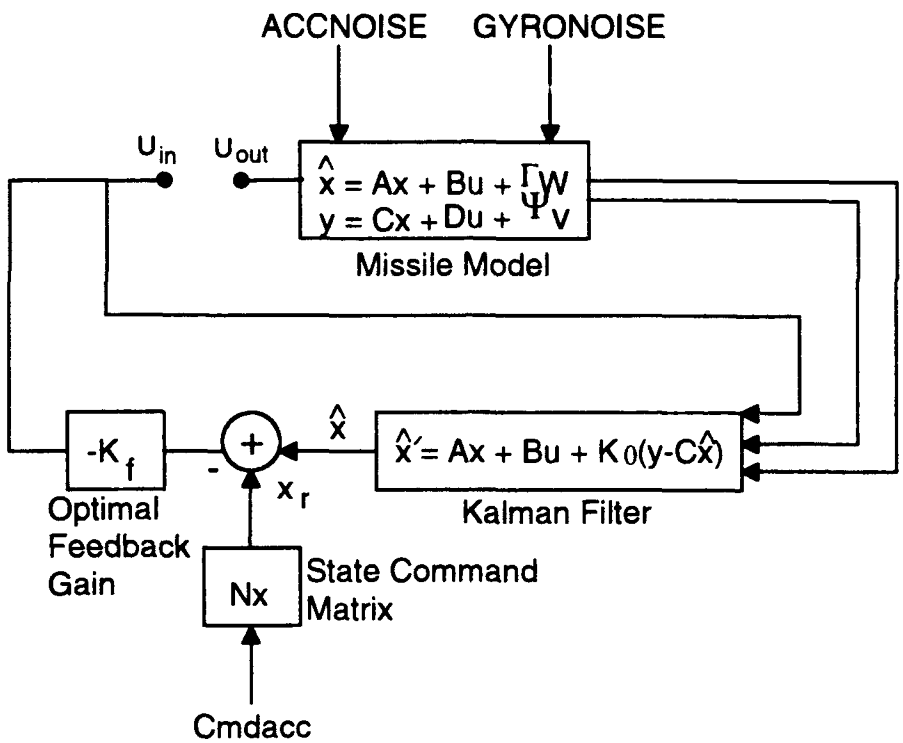


Figure 17. Optimal Linear Output Feedback Control System with Loop Broken at the Control Input.

It is important to realize that by adding an observer to the control system, the transfer function from the reference input to the reference output is not affected if the loop is broken at the control input. However, if the loop is broken anywhere else, the observer will affect the transfer function. In order to improve the robust stability of the LQG design proposed in Chapter III, loop transfer recovery will be applied by breaking the loop at the uncertainty, as shown in Figure 18.

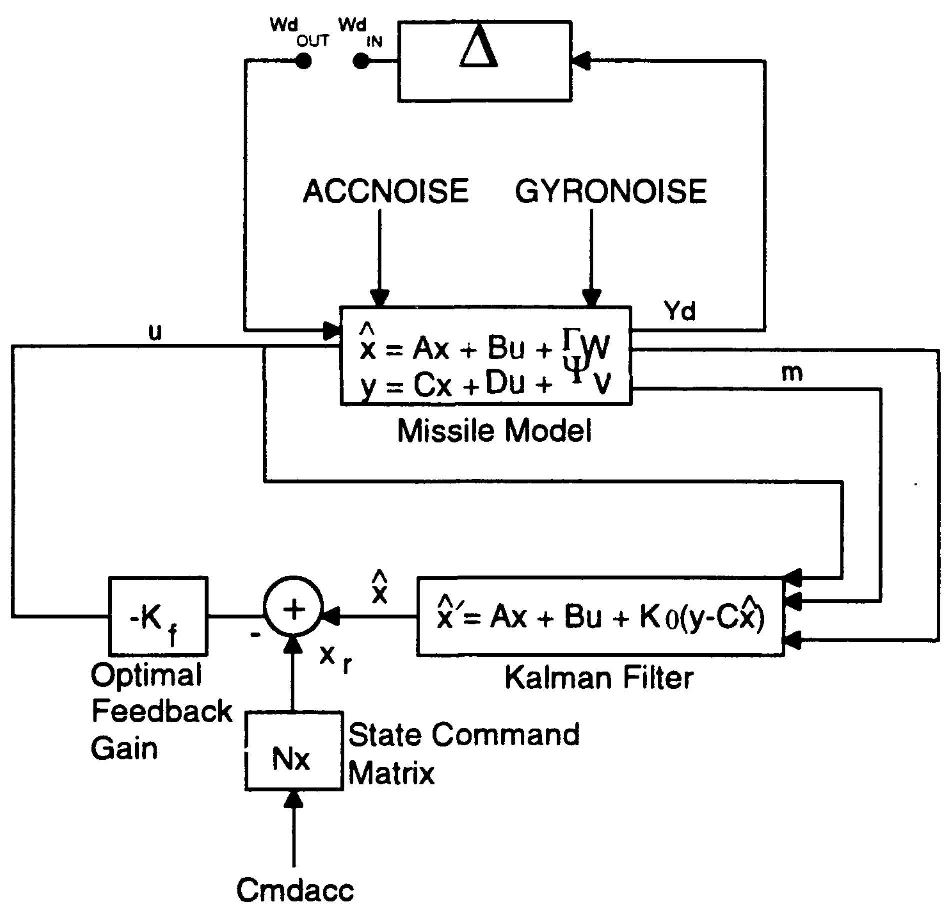


Figure 18. Optimal Linear Output Feedback Control System with Feedback Perturbation and Loop Broken at the uncertainty.

To make the design less sensitive to the perturbation Δ , fictitious noise will be added to the system at the uncertainty in the design process. Actually, fictitious noise is considered only in obtaining the optimal estimator gain, and is not present in the plant. The configuration shown in Figure 18 recalls the general Δ-P-K framework presented in Chapter II and illustrates this technique (see Figure 19).

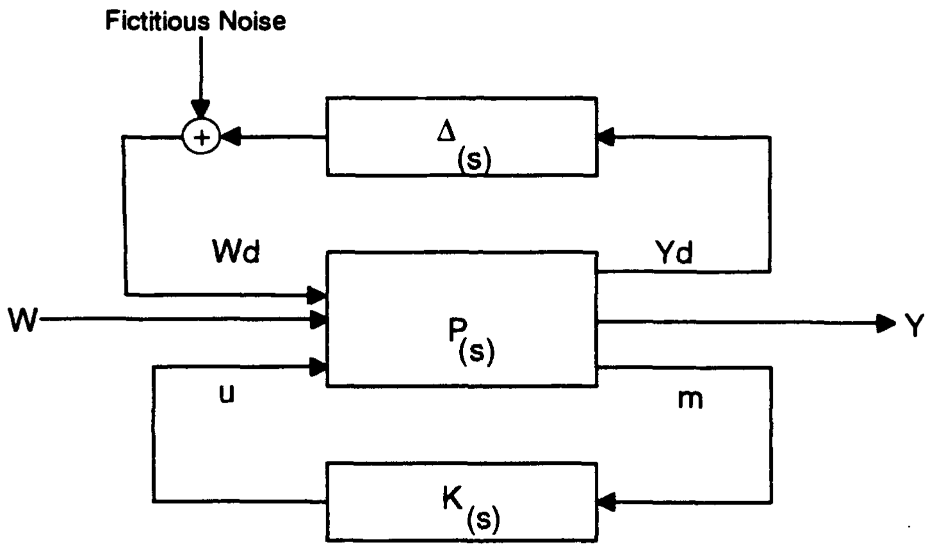
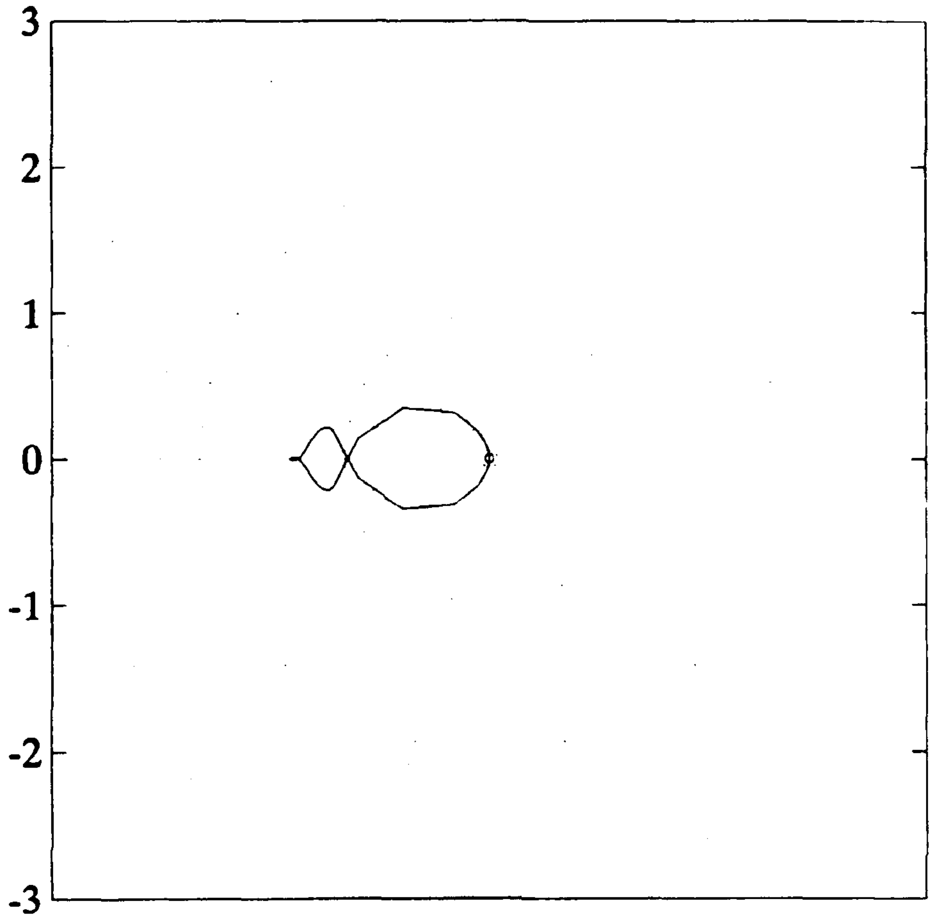


Figure 19. General Δ-P-K Framework with Fictitious Noise Added at the Uncertainty.

In this case the assumption of fictitious white noise, in addition to any white noise that may actually be present, enhances the robustness as the intensity of the fictitious noise tends to infinity asymptotically.

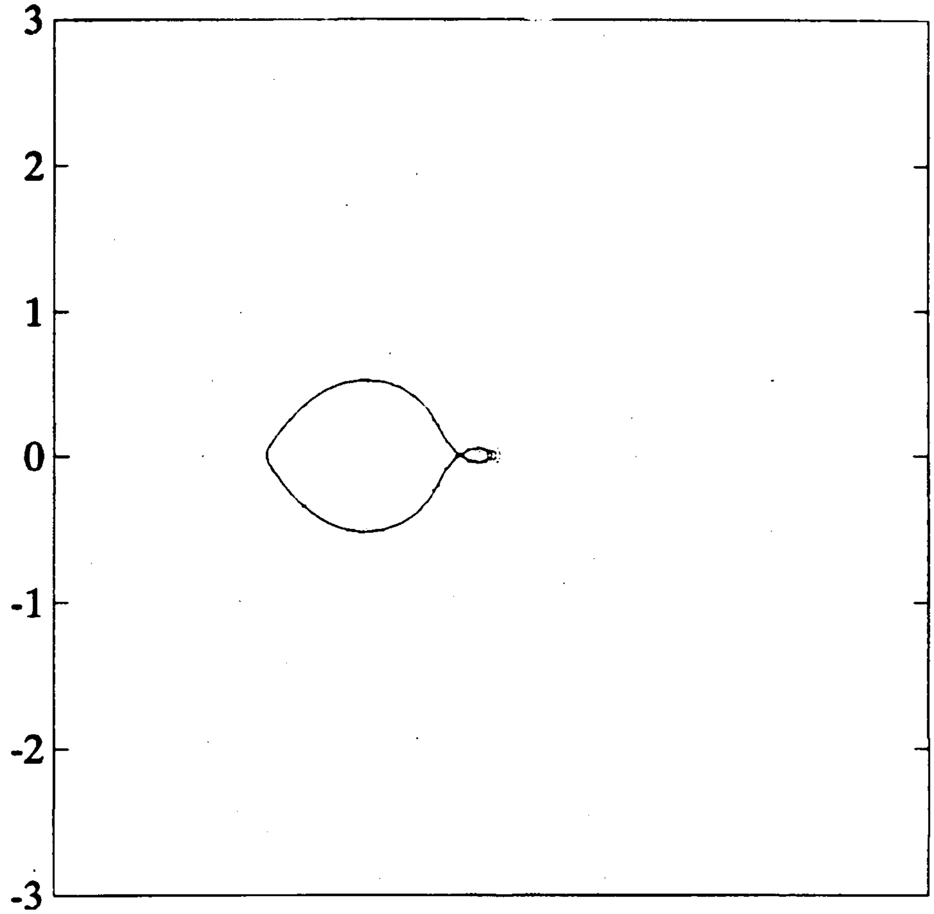
As the plant copes with the fictitious noise, the LQG controller becomes more robust to gain and phase plant input, but the optimality for the original nominal stochastic model is no longer guaranteed. In the case of non-minimum phase plants,$\sigma \to \infty$ will not lead to loop recovery. Thus, to recover open loop transfer functions with a series compensator, a plant **must be minimum phase.**

The following example includes a servo in parallel with the vibrational mode designed with LQG techniques. The sequence illustrates how the good robustness properties of an LQR controller (Gain Margin = ∞, Phase Margin $\geq 60°$, 6 dB margin against gain reductions) can be recovered by injecting fictitious noise at the control input.



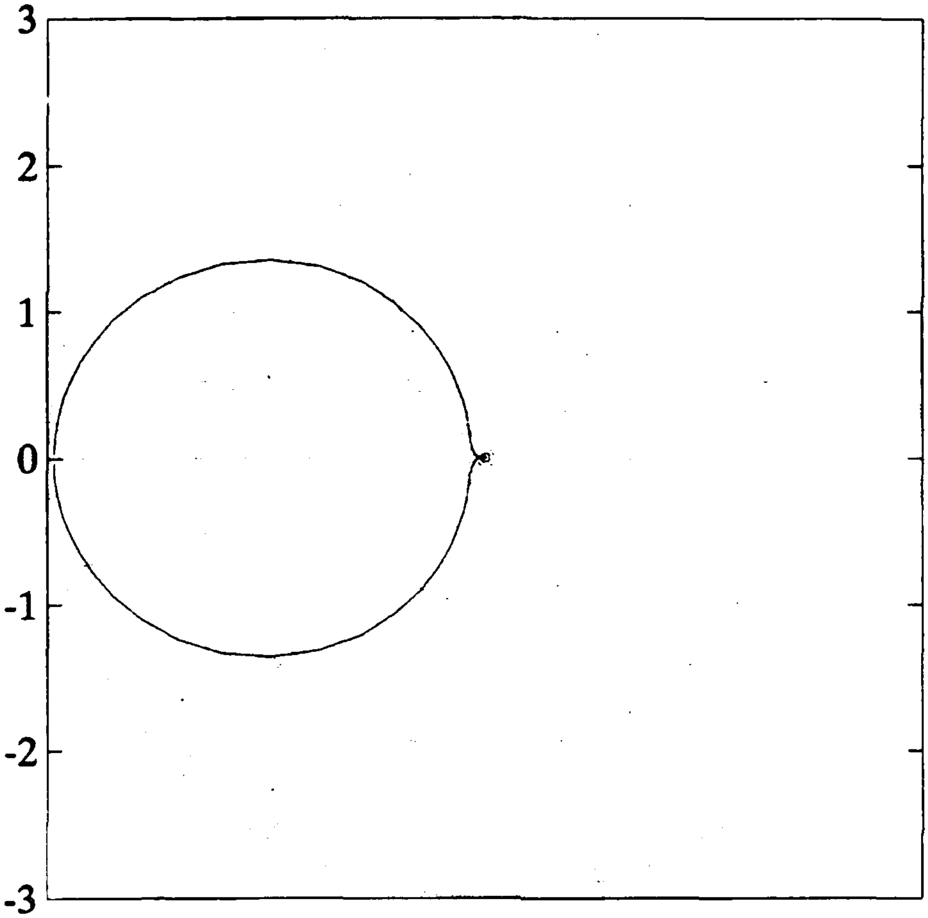
Gain Margin (GM)=1.11
Down-Side Gain Margin (DSGM)=0.72
Phase Margin (PM)=2.5

Figure 20. Nyquist Polar Plot for Optimal Control.



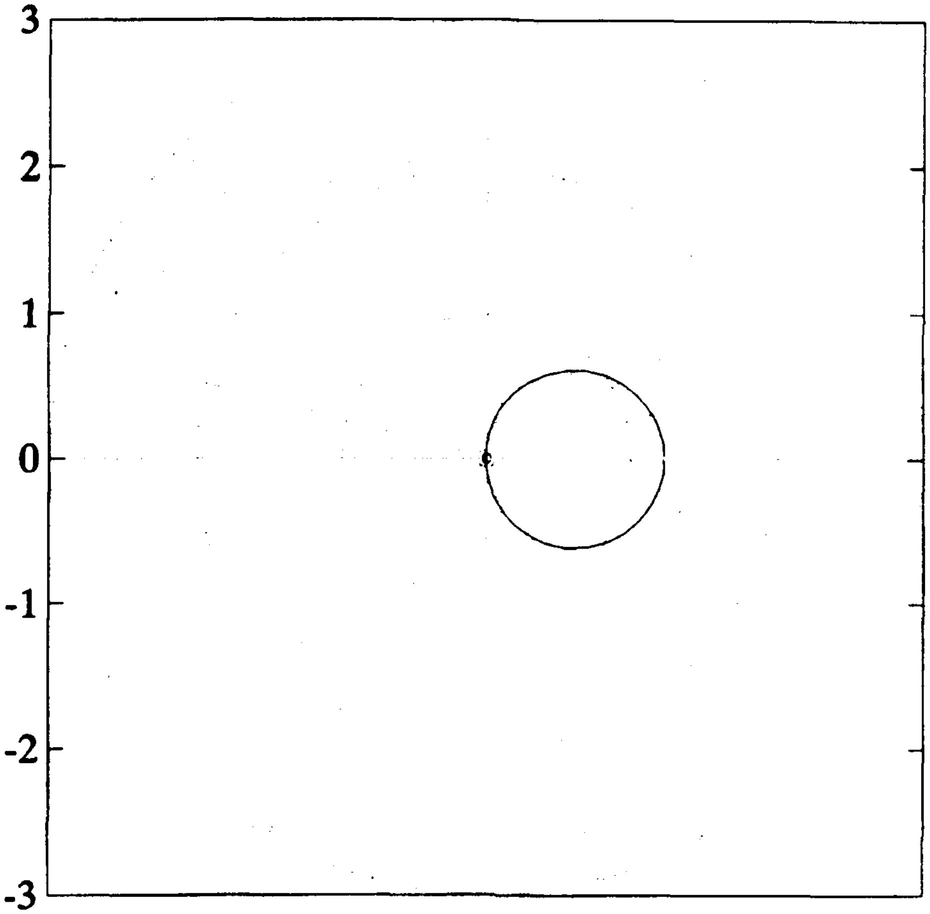
Variance of Fictitious Noise = 0.001
GM = 3.5
DSGM = .66
PM = 34.5

Figure 21. Nyquist Polar Plot for Optimal Control Plus Kalman Filter.



Variance of Fictitious Noise = 0.008
GM = 20
DSGM = .34
PM = 67.5

Figure 22. Nyquist Polar Plot for Optimal Control Plus Kalman Filter.



Variance of Fictitious Noise = 10
GM = ∞
DSGM = ∞
PM = 143

Figure 23. Nyquist Polar Plot for Optimal Control Plus Kalman Filter.

In the case of the Linear Quadratic Gaussian Controller Design for the flexible missile the uncertainties were not taken into account. As a result, Nominal Performance was achieved but Robust Performance and Robust stability were not. In other words, the

LQG Design becomes sensitive to parameter variation or uncertainties. The acceleration time response and the results of the μ–analysis test are presented in Figure 24.

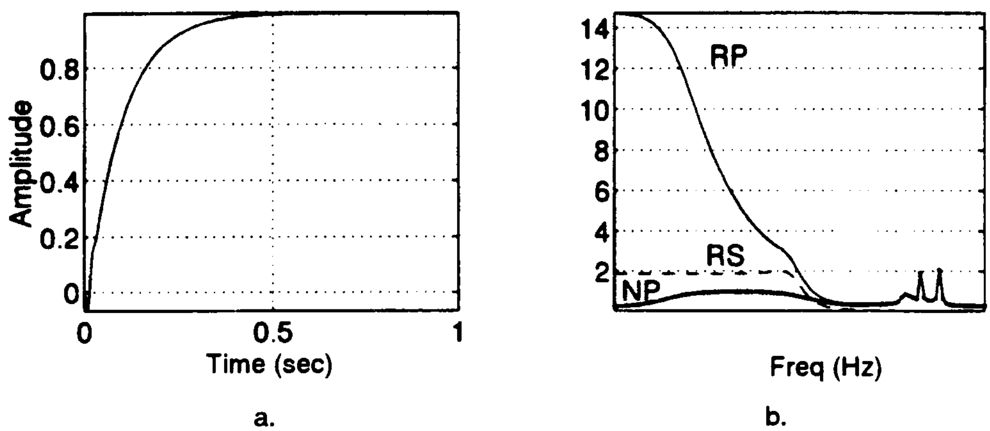
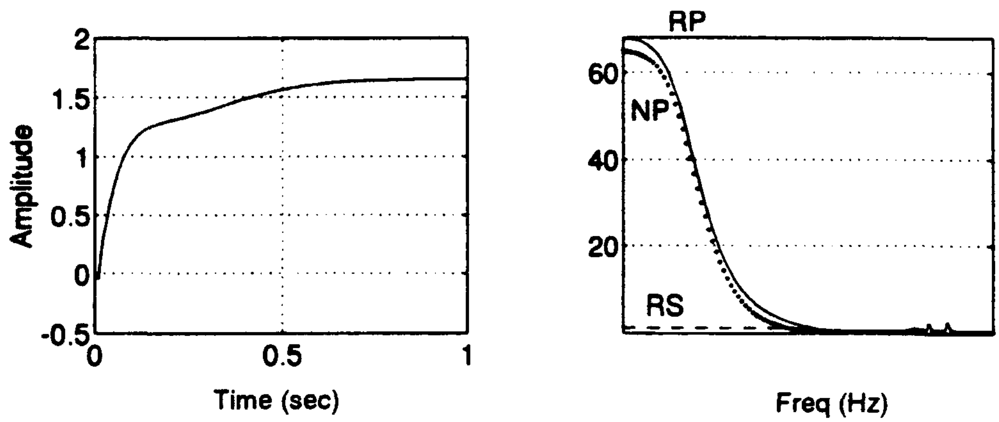


Figure 24. Acceleration Time Response (a) and μ-analysis (b) for LQG Design without Uncertainties.

The performance of the LQG design in the presence of fictitious noise injected at one or two uncertainties was evaluated. Loop Transfer Recovery through injection of fictitious noise at M_α in the presence of an uncertainty at Z_α is applied to enhance the robustness of the design. The effect of this technique on the time response and resulting μ-analysis tests are illustrated in Figures 25 through 28.

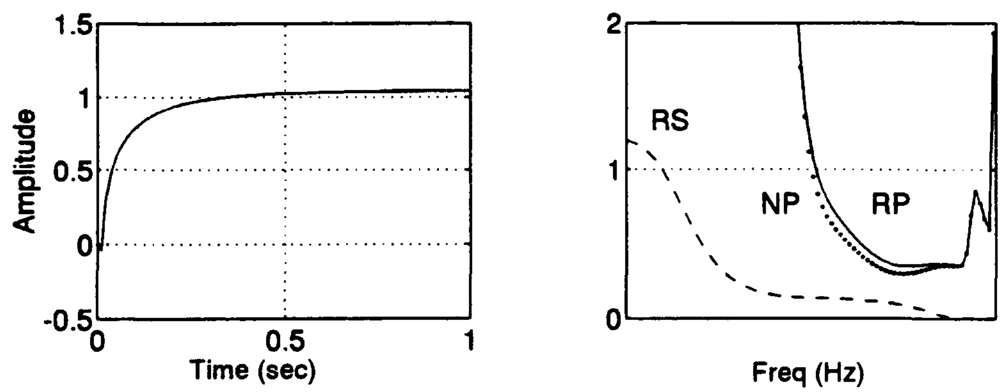
It can be seen in Figure 25(a) that once an uncertainty at Z_α is considered, the LQG controller is unable to track the desired reference input. In addition, robust performance, robust stability, and nominal performance requirements are not met (Figure 25(b)).



a. b.

Figure 25. Acceleration Time Response (a) and μ-analysis (b) for LQG Design in the Presence of an Uncertainty in Z_α.

As shown in Figure 26, time response is improved through the addition of fictitious noise at M_α, and μ-analysis results indicate a corresponding improvement in robust stability. Nominal performance and robust performance, however, are not met.



a. b.

Figure 26. Acceleration Time Response (a) and μ-analysis (b) for LQGLTR Design in the Presence of an Uncertainty in Z_α (Variance of the Fictitious Noise =1).

As the variance of the fictitious noise added at M_α increases, the acceleration time response of the system tends to track the desired reference input, and robust stability is achieved, as shown in Figures 27 and 28. Once again, neither robust performance nor nominal performance are met.

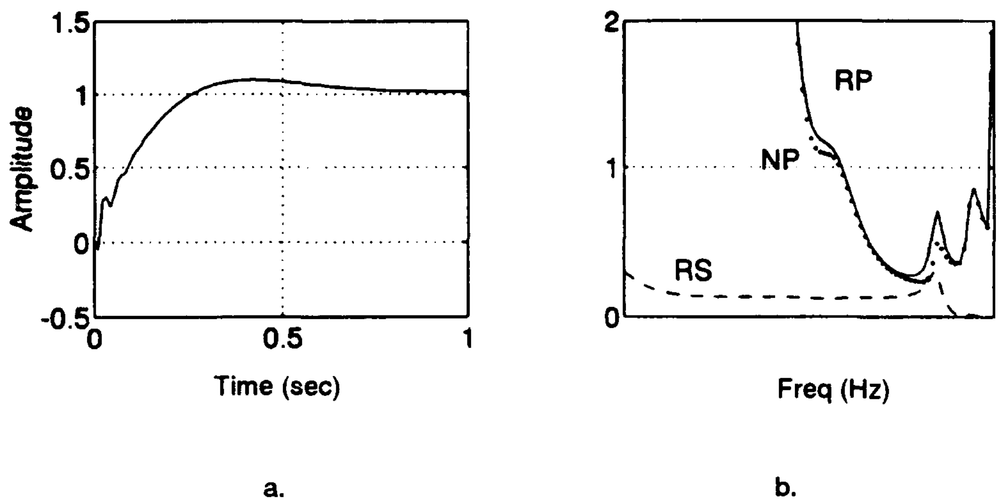
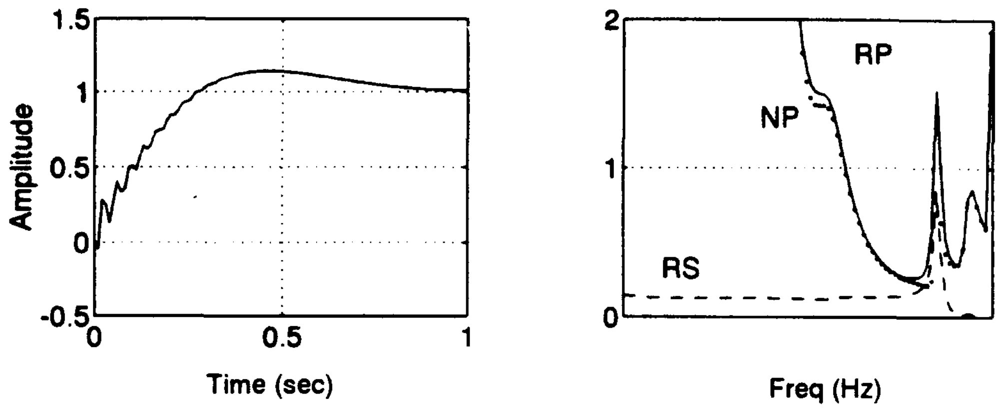


Figure 27. Acceleration Time Response (a) and μ-analysis (b) for LQGLTR Design in the Presence of an Uncertainty in Z_α (Variance of the Fictitious Noise=100).



a. b.

Figure 28. Acceleration Time Response (a) and μ-analysis (b) for LQGLTR Design in the Presence of an Uncertainty in Z_α (Variance of the Fictitious Noise=500).

So far, robust stability is achieved with enough intensity of the fictitious noise but as a consequence of loosing nominal performance. It must be recognized that there exists a tradeoff between performance loss for the nominal model and robustness gain. The simultaneous achievement of robust performance, robust stability, and nominal performance will require application of more sophisticated techniques, such as linear quadratic gaussian loop transfer recovery with formal synthesis and H_∞ optimal control, and is beyond the scope of the present work.

VI. CONCLUSIONS AND RECOMMENDATIONS

A. SUMMARY

This thesis presents the design of a controller for a flexible missile model using Linear Quadratic Gaussian techniques in combination with Loop Transfer Recovery in order to enhance the Robust Stability of the design.

A linearized model of the tail controlled surface-to-air missile was provided. The design of the Linear Quadratic Gaussian controller was based on a combination of the solution of the Stochastical Linear Quadratic Regulator (LQR) and Optimal Estimator problem (Kalman Filter). In order to transform the problem into a tracking problem a State Command Matrix (N_x) was designed to drive the reference input to a reference state. The Closed Loop Optimal Output Feedback Control System was then assembled and Robust Performance, Robust Stability, and Nominal Performance of the system was evaluated.

A Linear Quadratic Gaussian controller with Loop Transfer Recovery Techniques was then designed to improve the Robust Stability of the design. The μ–Analysis and Synthesis Toolbox and the Control Toolbox of MATLAB were used for the assembling, design, analysis and simulation of the missile system.

B. CONCLUSIONS

In this study, a Linear Quadratic Gausian Controller is developed for the model of a tail controlled surface-to-air missile without taking in consideration the uncertainties present in the model. All of the required Time Domain Specifications for the missile autopilot are met by the LQG design. The controller does not saturate the tail deflection actuator rate capabilities nor destabilize the high frequency flexible body modes of the missile. Nominal Performance is met by the LQG design, but Robust Performance and Robust Stability are not met. In short, the LQG controller meets the design goal at the nominal operating condition but it becomes sensitive with parameter variation. Some linear combination of the uncertainties might destabilize the system.

To improve the Robust Stability of the system, Loop Transfer Recovery is then applied to the LQG design by the injection of fictitious noise at different uncertainties. All of the required Time Domain Specifications for the missile autopilot are met by the LQGLTR design. The controller does not saturate the tail deflection actuator rate capabilities nor destabilize the high frequency flexible body modes of the missile. Robust Stability is also met but as a consequence of loosing Nominal Performance. Robust Performance is not met. A trade-off must be observed between the loss of nominal Performance and the gain of Stability Robustness.

C. RECOMMENDATIONS

The LQG controller designed in this work is not recommended for the missile model in question since it is sensitive under parameter variations or uncertainties which could destabilize the system.

The LQGLTR controller designed in this work can be used for the missile model in question if Robust Stability is desired and the designer is willing to make a trade-off in the nominal performance of the system.

The use of more sophisticated techniques such as H_∞ Optimal control and μ-synthesis may allow the designer to meet the design specifications as well as all of the performance tests.

LIST OF REFERENCES

1. Bibel, John E. and Stalford, Harold L., "An Improved Gain-Stabilized μ Controller for a Flexible Missile," AIAA-92-0206, *30th Aerospace Sciences Meeting*, 1992.

2. Wise, Kevin and Mears, Barry C., "Missile Autopilot Design Using H_∞ Optimal Control with μ-Synthesis," AIAA-92-0418, 1992.

3. Burl, Jeff, "Robust Multivariable Control," Class Notes, Naval Postgraduate School Department of Electrical and Computer Engineering, 1992.

4. Bibel, John E. and Stalford, Harold L., "μ-Synthesis Autopilot Design for a Flexible Missile," AIAA-91-0586, *29th Aerospace Sciences Meeting*, 1992.

5. Rajnikant, Patel V., and Munro, Neil, *Multivariate System Theory and Design*, Pergamon Press, 1982.

6. Saberi, Ali, Chen, Ben M., and Peddapullaiah, Sannuti, *Loop Transfer recovery Analysis and Design*, Springer-Verlag, 1993.

INITIAL DISTRIBUTION LIST

1.	Defense Technical Information Center Cameron Station Alexandria, VA 22304-6145	2
2.	Library, Code 52 Naval Postgraduate School Monterey, CA 93943-5002	2
3.	Chairman, Code EC Department of Electrical and Computer Engineering Naval Postgraduate School Monterey, CA 93943-5002	1
4.	Professor Roberto Cristi, Code EC/Cx Department of Electrical and Computer Engineering Naval Postgraduate School Monterey, CA 93943-5002	2
5.	Professor Harold Titus, Code EC/Ts Department of Electrical and Computer Engineering Naval Postgraduate School Monterey, CA 93943-5002	1
6.	Professor George Thaler, Code EC/Ta Department of Electrical and Computer Engineering Naval Postgraduate School Monterey, CA 93943-5002	1
7.	Professor Jeff Burl Department of Electrical Engineering Michigan Technological University 1400 Townsend Drive Houghton, MI 49931-1295	1

8. Lieutenant Fernando Jiménez 2
   Canopus A-1
   Surco Lima 33
   Perú